Developments in Axisymmetric Gravity

When a subject is too complex, it is most natural to reduce it by imposing restrictions, delimit smaller areas, and then remove them progressively until the whole is understood. The idea is to design a division in parts of similar difficulty but one that we are capable of analyzing. All at once the whole may be difficult but not when we think it in appropriate sectors. This reductionism occurs most often spontaneously when the subject is too big that there is no agreement or simply no possibility of agreement among scientists on how to make a convenient partition. Under such circumstances people from different backgrounds, different places and sometimes at different epochs, chose what they believe is more interesting and more convenient to do. The subject is then investigated a bit chaotically, what carries obvious advantages and disadvantages. Axisymmetric Gravity enjoys a bit of everything.

Celestial bodies with the simplest shapes are those spherically symmetric. Such objects look the same when we rotate them in any direction. Axisymmetric bodies instead have the next degree of complexity. They look the same when we rotate them only through a particular axis, the *symmetry axis*. A soccer ball is spherically symmetric, but pears or eggs are just axially symmetric (roughly). Axisymmetric gravitational systems are, by definition, ones having an axis of symmetry. They can be for instance stars, pulsars, galaxies, black holes or gravitational waves. These systems are obviously important in the current international context of research. Its complexity balances somewhere between that of the whole Einstein's theory and that of very reduced systems considered in the past.

Historically, mathematical studies of General Relativity were divided according to their mathematical complexity (this can be good or bad, but it is undeniably human). It turns out that the more symmetries a system has, the simpler it is to analyze. For this reason spherically symmetric systems are among the simplest ones and the more investigated. On the other hand axisymmetric systems are more complex and require also greater mathematical complexity. Let us bring some examples. Spherically symmetric (vacuum) black holes there are only of one type, the *Schwarzschild black holes* and are parametrized simply by their mass. They were discovered by K. Schwarzschild in 1915 just a few months after Einstein's explanation of Mercury's perihelion and before the definitive version of General Relativity appeared (!). For many reasons, Schwarzschild's article is a historical landmark. But it took almost fifty years until rotating axisymmetric black holes were found by R. Kerr in 1963 and are parametrized by their mass and angular momentum. Kerr's achievement is another landmark (here it does not matter how mathematically simple things look from our contemporary eyes).

The story ends with the proof that stationary-axisymmetric single black holes are known to be just of the Kerr type. This is the result of many years of research, with many highlights, but I will not enter into that here.

But what about two black holes aligned along a symmetry axis (as depicted in Figure 1)? If two aligned black holes were to spin in opposite directions, the net effect of the counter rotation would be a net repulsion. Could then two black holes remain each other apart (but in equilibrium) by the balance between spin repulsion and gravitational attraction? And, what about three or more holes in equilibrium? This is an interesting story in which many people at the AEI played a role. It is also a story that has more to say and this is why we are describing it in this short report.

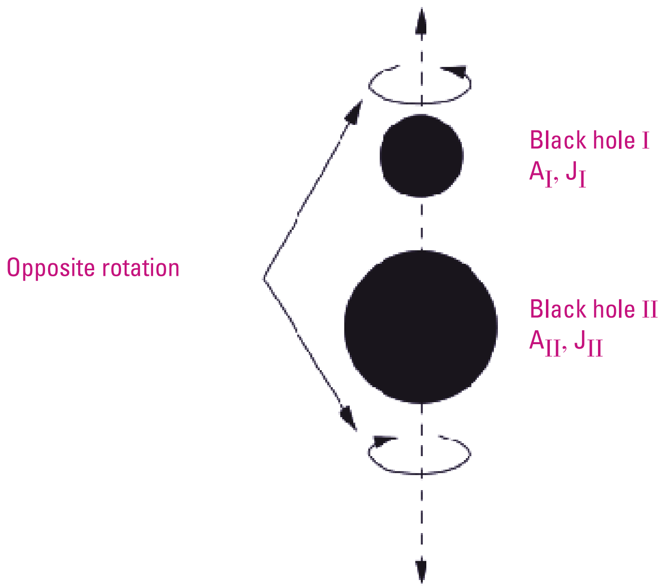

Fig.1: Representation of two aligned rotating black holes of areas A_1 and A_2 and angular momentums J_1 and J_2.

What Gernot Neugebauer (F. Schiller University - Jena) and Jörg Hennig (Otago University - New Zealand; formerly at the AEI) had found was that such configuration in fact cannot exist (see [2] and references therein). The key ingredient was the remarkable inequality $A \geq 8\pi|J|$ between the area A and the angular momentum $|J|$ of black holes. The inequality roughly says that the more black holes rotate the bigger they are. What Neugebauer and Hennig showed was that, if two aligned black-holes exist in equilibrium, then the inequality $A \geq 8\pi|J|$ cannot hold simultaneously at both holes. They concluded then, a posteriori, that the assumed configuration is impossible. The inequality $A \geq 8\pi|J|$ was first proved by Hennig, Ansorg and Cederbaum (all formerly at AEI) for stationary black-holes. What is remarkable is that it is also an inequality valid for dynamical (i.e. non-stationary) axisymmetric black holes as shown by S. Dain and the author [1]. As important, the method of proof in [1] allowed a variety of other achievements. First, it was extended to higher dimensions (by Hollands), extended to include charge (by Gabach, Jaramillo & Reiris), and to include a dilaton field (by Yazadjiev). Finally, in a joint work (to appear) between the author and Eugenia Gabach (formerly at the AEI) it allowed a complete description of the whole geometry of dynamical black holes, namely it allowed a detailed and accurate account of the shapes that black holes can enjoy.

Many avenues of research are open for the time to come. First, it is still open the question whether more than two black holes could stay in axisymmetric equilibrium. Or, what about non-aligned black holes?

References

[1] S. Dain and M. Reiris, Area-Angular momentum inequality for axisymmetric black-holes. *Phys. Rev. Lett.* **107**:051101 (2011).

[2] G. Neugebauer and J. Hennig, Stationary two black-hole configurations: a non existence proof. *J. Geom. Phys.*, **62**:613-630, 2012.

Could there exist configurations where several non-aligned black holes remain in equilibrium? In other words are the Kerr black holes the only (vacuum) ones existing in nature? These old questions entangle fascinating mathematical difficulties that are currently being investigated by the community and where the developments mentioned above could play a big role.

Martin Reiris

From Here to Infinity on a Single Computer

Numerical relativity has made great strides in recent years. Simulations of binary black hole mergers, a major unsolved problem until the breakthrough in 2005, have by now become routine. Yet several issues remain, and these can often benefit from an improved mathematical understanding of the field equations and global properties of their solutions. In the following we give one example of such a problem.

A reasonable idealisation of a common situation in astrophysics is an *isolated system*, i.e. an asymptotically flat spacetime containing a compact self-gravitating source, e.g. a neutron star, black hole binary, etc. The highly non-linear dynamics close to the source require a numerical solution of the field equations. But how do we represent the entire unbounded domain with finite computational resources?

Let us recall that there are three different types of "infinity" in general relativity: spatial infinity, future/past timelike infinity (which the worldlines of observers approach) and future/past null infinity (which light rays approach). This can be conveniently represented in a Penrose diagram (Figure 1).

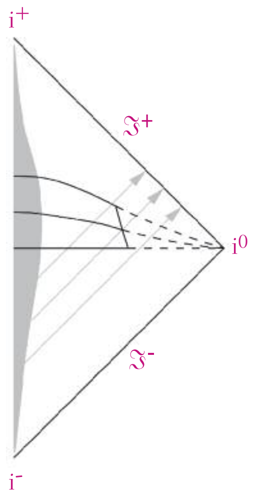

Fig.1: Penrose diagram of Minkowski space, with spatial infinity i^0, future/past timelike infinity $i^\pm$ and future/past null infinity $\mathfrak{I}^\pm$. Included is a compact source (grey area) and outgoing radiation (grey arrows). The horizontal lines represent a foliation of spacetime into spacelike hypersurfaces approaching spatial infinity, truncated at a finite distance (vertical line).

Evolution with finite boundary

The standard method is to slice spacetime into spacelike hypersurfaces approaching spatial infinity (also shown in Figure 1). Each slice corresponds to one instant of time. In the 3+1 formulation of general relativity, the Einstein equations split into *constraint equations* that must hold on each slice, and *evolution equations* that take us from one slice to the next. As is apparent from Figure 1, outgoing radiation never leaves the spatial slices as time proceeds, because all the slices end at spatial infinity. Therefore, compactifying the spatial coordinates on the slices so that spatial infinity is brought to a finite coordinate location is not a good idea because the wavelength of the radiation will appear to be increasingly "blue-shifted" and ultimately will fail to be resolved on the numerical grid.

Instead, one usually truncates the spatial slices at a finite distance from the source and only solves the equations in the interior. This introduces an artificial timelike boundary, where boundary conditions must be imposed. Among other things, these should (ideally) guarantee that the solution on the truncated domain is identical with the solution on the unbounded domain. In particular, one would like